Mathematical Modelling of Long-Wave for Interfacial Waves in Two-Layer Fluids Based on the Dirichlet-Neumann Operator

Hanxi Wei[1,a,*]

[1]The Experimental High School Attached to Beijing Normal University, Beijing, China

[a]weihanxi2008@outlook.com

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Abstract: This study investigates the long-wave dynamics of interfacial waves in a two-layer incompressible, inviscid fluid system. Starting from the full potential formulation, we employ a Dirichlet–Neumann operator approach to derive a new class of nonlinear evolution equations, thereby avoiding the direct solution of Laplace's equation. By performing asymptotic expansions of the Dirichlet–Neumann operator under various long-wave scaling, we obtain a hierarchy of reduced models—including the Korteweg–de Vries (KdV), fifth-order KdV, modified KdV (mKdV), and Benjamin equations. This approach significantly reduces computational complexity while retaining key nonlinear and dispersive effects, and provides a unified and efficient framework for modelling interfacial wave propagation in stratified fluids.

1. Introduction

This paper investigates the mathematical modelling of interfacial waves in a two-layer fluid system. Interfacial waves, which commonly occur at the boundary between two fluid layers with different densities, such as ocean layers of varying salinity and temperature, are significant in both scientific and practical contexts [1] . These waves, often referred to as internal waves, play a crucial role in ocean dynamics, influencing mixing, nutrient transport, and even climate patterns.

The mathematical modelling and well-posedness of interfacial waves in two-layer fluids have been extensively studied. Early models, like the Korteweg-de Vries (KdV) equation and Benjamin-Ono equation, relied on the assumptions of weak nonlinearity and weak dispersion. Experiments have shown that the KdV equation is widely applicable, particularly for long-wave approximations in interfacial waves [2]. For nonlinear waves in deep waters, the KdV equation is even superior to the Benjamin-Ono equation [3]. However, the KdV model breaks down when wave amplitudes grow large, and nonlinear effects become more significant, violating the weakly nonlinear assumption of the kdv framework [4]. Phenomena such as broad wave platforms and conjugate flows, observed in experiments and oceanographic studies [5], are beyond the scope of the KdV equation. The modified KdV (mKdV) equation, introduced by [6], resolves these issues, effectively modelling such structures. In cases where capillarity becomes important, the Benjamin equation [7] predicts wave-packet solitary with decaying oscillatory tails, a phenomenon numerically computed and analyzed by [8] and predicted theoretically by [9] using a fifth-order KdV equation. The wave-packet solitary, bifurcate from periodic waves with infinitesimally small amplitudes and are characterized by the nonlinear Schrödinger equation in the small amplitude regime[10].

However, the derivation of these model equations is based on traditional asymptotic analysis methods, which unavoidably involve solving the Laplace equation, making the derivation process cumbersome and computationally expensive. One way to simplify the computation is to avoid directly solving the Laplace equation. Zakharov made a significant advancement by choosing energy as the Hamiltonian and using wave height and surface potential as canonical variables [11]. This demonstrated that the water wave system in a single fluid layer can be treated as a Hamiltonian system. However, this method still involves solving the Laplace equation and applying boundary conditions. Later, Craig and Sulem expanded the Dirichlet-Neumann operator using a Taylor series and reformulated the kinematic and dynamic boundary conditions in terms of the canonical variables [12].

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This approach avoids solving the Laplace equation directly, thereby reducing computational effort.

In the present work, we follow this approach and derive a hierarchy of nonlinear model equations for interfacial waves using expansions of the Dirichlet–Neumann operator. By combining asymptotic analysis with this operator framework, we obtain a range of long-wave models, including the KdV equation, fifth-order KdV equation, modified KdV equation, and Benjamin equations.

2. Mathematical Formulation

We consider two mutually incompressible, ideal, inviscid fluids as shown in Figure 1. The fluids are bounded by solid walls above and below. When the fluids are at rest, the thicknesses and densities of the upper and lower layers are denoted as $h^{\pm}$ and $\rho^{\pm}$, where the superscripts $+$ and $-$ refer to the upper and lower layers, respectively. We establish a Cartesian coordinate system with the y-direction aligned with the opposite direction of gravity. The interface between the two fluid layers when at rest is located at $y = 0$, and the x-direction is horizontal. We examine the irrotational flow of the fluids; indeed, for ideal, inviscid, and incompressible fluids, irrotationality is preserved as long as it is initially present, in accordance with Helmholtz's theorem[13]. Consequently, the flow is potential, and the potential functions $\phi^{\pm}$ for the upper and lower layers of fluid satisfy Laplace's equation:

$$\phi^+_{xx} + \phi^+_{yy} = 0,\ \eta < y < h^+, \tag{1}$$

$$\phi^-_{xx} + \phi^-_{yy} = 0,\ -h^- < y < \eta, \tag{2}$$

Where, $\eta = \eta(x, t)$ represents the shape of the interface between the two fluid layers. At the interface between the two fluid layers, the kinematic and dynamic boundary conditions are satisfied as follows:

$$\eta_t = \phi^+_y - \eta_x \phi^+_x,\ y = \eta(x,t), \tag{3}$$

$$\eta_t = \phi^-_y - \eta_x \phi^-_x,\ y = \eta(x,t), \tag{4}$$

$$\rho^-\left[\phi^-_t + \frac{1}{2}\left|\nabla\phi^-\right|^2 + g\eta\right] - \rho^+\left[\phi^+_t + \frac{1}{2}\left|\nabla\phi^+\right|^2 + g\eta\right] - \frac{\sigma\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}} = 0,\ y = \eta(x,t), \tag{5}$$

Where, $\nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is gradient operator, σ denotes the surface tension coefficient. At the fixed wall boundaries of the upper and lower layers, the no-penetration boundary condition is satisfied:

$$\frac{\partial\phi^{\pm}}{\partial y} = 0,\ y = \pm h^{\pm}. \tag{6}$$

We define the surface potential functions for the upper and lower fluid layers as $\xi^{\pm} = \phi^{\pm}(x, \eta(x,t), t)$ and introduce the Dirichlet–Neumann (DtN) operators corresponding to each layer. For the lower fluid, the DtN operator is given by

$$G^-\xi^- = \left(\phi^-_y - \eta_x\phi^-_x\right)|_{y=\eta(x,t)} = \frac{\partial\phi^-}{\partial n}\sqrt{1+\eta_x^2}, \tag{7}$$

And for the upper fluid,

$$G^+\xi^+ = \left(\eta_x\phi^+_x - \phi^+_y\right)|_{y=\eta(x,t)} = -\frac{\partial\phi^+}{\partial n}\sqrt{1+\eta_x^2}. \tag{8}$$

Thus the kinematic boundary conditions (30) can be rewritten as

$$\eta_t = -G^+\xi^+ = G^-\xi^- = G^-\left(\rho^- G^+ + \rho^+ G^-\right)^{-1} G^+\xi, \tag{9}$$

Where $\xi = \rho^- \xi^- - \rho^+ \xi^+$. For the DtN operator, if η is less than a certain value, the DtN operator is analytic, which also means we can perform a Taylor expansion of the DtN operator[12]. Following the method of Craig[12], we can expand the operator G^- and G^+

$$\begin{aligned} &G^- = G_0^- + G_1^- + ..., \\ &G_0^- = D\tanh\left(h^- D\right), \\ &G_1^- = D\eta D - D\tanh\left(h^- D\right)\eta\tanh\left(h^- D\right)D, \end{aligned} \tag{10}$$

And

$$\begin{aligned} &G^+ = G_0^+ + G_1^+ + ..., \\ &G_0^+ = D\tanh\left(h^+ D\right), \\ &G_1^+ = -D\eta D + D\tanh\left(h^+ D\right)\eta\tanh\left(h^+ D\right)D, \end{aligned} \tag{11}$$

Where, $D = -\mathrm{i}\partial_x$, it is important to note that the DtN operator is actually a pseudodifferential operator.

The dynamic boundary condition may equivalently be written as

$$\begin{aligned} \rho^-\left[\xi_t^- + \frac{1}{2}\left(\xi_x^-\right)^2 - \frac{\left(\eta_t + \eta_x\xi_x^-\right)^2}{2\left(1+\eta_x^2\right)}\right] - \rho^+\left[\xi_t^+ + \frac{1}{2}\left(\xi_x^+\right)^2 - \frac{\left(\eta_t + \eta_x\xi_x^+\right)^2}{2\left(1+\eta_x^2\right)}\right] \\ + \left(\rho^- - \rho^+\right)\eta - \frac{\sigma\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}} = 0. \end{aligned} \tag{12}$$

Without loss of generality, we select h^-, $\sqrt{gh^-}$, and $\sqrt{g(h^-)^3}$ as the characteristic length, characteristic velocity, and characteristic potential function, respectively. We introduce the following dimensionless quantities:

$$h = \frac{h^+}{h^-}, \rho = \frac{\rho^+}{\rho^-}, B = \frac{\sigma}{\rho^- g\left(h^-\right)^2}, \tag{13}$$

The three dimensionless quantities introduced are the ratio of the depths of the two fluid layers, the ratio of the densities of the two fluid layers, and the Bond number, which represents the ratio of capillary forces to gravitational forces. Consequently, the dimensionless kinematic and dynamic boundary conditions can be expressed as follows:

$$\begin{aligned} &\eta_t = G^-\left(G^+ + \rho G^-\right)^{-1} G^+\xi, \\ &\xi_t + \frac{1}{2}\left[\left(\xi_x^-\right)^2 - \frac{\left(\eta_t + \eta_x\xi_x^-\right)^2}{\left(1+\eta_x^2\right)}\right] - \frac{\rho}{2}\left[\left(\xi_x^+\right)^2 - \frac{\left(\eta_t + \eta_x\xi_x^+\right)^2}{\left(1+\eta_x^2\right)}\right] + \left(1-\rho\right)\eta - \frac{B\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}} = 0, \\ &\xi^- = \left(G^+ + \rho G^-\right)^{-1} G^+\xi, \\ &\xi^+ = -\left(G^+ + \rho G^-\right)^{-1} G^-\xi, \\ &\xi = \xi^- - \rho\xi^+. \end{aligned} \tag{14}$$

3. Nonlinear Long-Wave Model

In this section, the Dirichlet-Neumann operator expansion method will be employed to derive a series of nonlinear long-wave models. The derivation assumes a long-wave approximation where the characteristic wavelength is substantially larger than the depth of the lower fluid layer. This assumption is quantified by the parameter $\mu = h^{-}/\lambda \ll 1$ where h^{-} is the depth of the lower fluid layer and λ is the characteristic wavelength.

3.1. Shallow (Lower Layer) – Shallow (Upper Layer) Model

Introducing the long wave parameter $\mu = h^{-}/\lambda \ll 1$ and assuming that the depth ratio $h = O(1)$, these two assumptions imply that the depth of the two fluid layers is of the same order of magnitude, while the wavelength is much larger than the depth of the fluid layers. We first consider the classical Boussinesq scaling:

$$\partial_x = O(\mu), \partial_t = O(\mu), \eta = O(\mu^2), \xi = O(\mu), \rho = O(1), \tag{15}$$

Thus, by asymptotically expanding the operator $G^{\pm}$ in terms of the small parameter μ, and omitting the detailed mathematical derivations, we obtain:

$$\begin{aligned} G^{-} &= -\partial_{xx} - \frac{1}{3}\partial_{xxxx} - \partial_x \eta \partial_x + O(\mu^6), \\ G^{+} &= -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} + \partial_x \eta \partial_x + O(\mu^6), \\ \left(G^{+} + \rho G^{-}\right)^{-1} &= -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3(h+\rho)^2} - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x^{-1} + O(\mu^2). \end{aligned} \tag{16}$$

By substituting equation (16) into (14), and neglecting higher-order terms, we obtain:

$$\eta_t + \frac{h}{h+\rho}\xi_{xx} + \frac{h^2(1+h\rho)}{3(h+\rho)^2}\xi_{xxxx} + \frac{h^2-\rho}{(h+\rho)^2}\partial_x(\eta\xi_x) = 0, \tag{17}$$

$$\xi_t + (1-\rho)\eta - B\eta_{xx} + \frac{h^2-\rho}{2(h+\rho)^2}\xi_x^2 = 0. \tag{18}$$

By combining equations (73) and (74), and eliminating η, we obtain:

$$\xi_{tt} - \frac{h(1-\rho)}{h+\rho}\xi_{xx} + \frac{h}{h+\rho}\left[B - \frac{(h-h\rho)(1+h\rho)}{3(h+\rho)}\right]\xi_{xxxx} + \frac{h^2-\rho}{2(1+\rho)^2}\partial_t\xi_x^2 + \frac{h^2-\rho}{(h+\rho)^2}\partial_x(\xi_t\xi_x) = 0. \tag{19}$$

We first examine the coefficients of the linear term ξ_{xx} and the quartic term ξ_{xxxx}:

$$c^2 := \frac{h(1-\rho)}{h+\rho}, \alpha := \frac{h}{h+\rho}\left[B - \frac{(h-h\rho)(1+h\rho)}{3(h+\rho)}\right], \tag{20}$$

By performing a Taylor expansion of the dispersion relation (\ref{dispersion}) under the specified scaling, we find that the first two terms of the expansion align with those in the above expression. Subsequently, introducing the coordinate transformation $X = x - ct$, the new variable $\tau = \mu^3 t$, and setting $H = \xi_X$, we obtain the well-known Korteweg-de Vries (KdV) equation from equation (19):

$$H_\tau - \frac{\alpha}{2c}H_{XXX} + \frac{3(h^2-\rho)}{2(h+\rho)}HH_X = 0. \tag{21}$$

From equation (21), it can also be deduced that:

$$\eta = A\,\mathrm{sech}^2\left\{\sqrt{\frac{\alpha h(h+\rho)}{36c^2(\rho-h^2)}}A\left[X + \frac{\alpha}{6c}A\tau\right]\right\}, \tag{22}$$

Thus, the KdV model for wave height in the case of shallow water-shallow water has been derived. It is important to note that the coefficient α in the third-order dispersion term of equation (21) can potentially be much smaller than 1. In such scenarios, the asymptotic expansion method may no longer be valid. Therefore, we need to choose new scaling relations:

$$\partial_x = O(\mu), \partial_t = O(\mu), \eta = O(\mu^2), \xi = O(\mu^3), \rho = O(1), \alpha = O(\mu^2). \tag{23}$$

Through a derivation similar to that of the KdV equation, we obtain the fifth-order KdV equation [14,15]:

$$\eta_\tau - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_X + \frac{\beta}{2c}\eta_{XXXXX} = 0, \tag{24}$$

Where,

$$\beta = \frac{h}{h+\rho}\left[\frac{2h^3(1+h^3\rho)}{15}c^2 - \frac{\rho(1-h^2)^2}{9(h+\rho)}c^2 - \frac{h(1+h\rho)}{3(h+1)^2}B\right]. \tag{25}$$

In addition to the fact that the coefficient α of the third-order dispersion term in equations (21) might be much smaller than 1, the coefficient of the nonlinear term can also be significantly less than 1. In such cases, the original asymptotic expansion method and scaling used for deriving the KdV equation are no longer valid. Therefore, a new scaling relationship must be adopted [14].

$$\partial_x = O(\mu), \partial_t = O(\mu), \eta = O(\mu), \xi = O(1), \rho = O(1), \frac{3c(h^2-\rho)}{2h(h+\rho)} = O(\mu). \tag{26}$$

Unlike the derivation of the fifth-order KdV equation, here we need to re-expand the Dirichlet-Neumann operator asymptotically:

$$\begin{aligned}
G^- &= -\partial_{xx} - \frac{1}{3}\partial_{xxxx} - \partial_x\eta\partial_x + O(\mu^5), \\
G^+ &= -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} + \partial_x\eta\partial_x + O(\mu^5), \\
(G^+ + \rho G^-)^{-1} &= -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3(h+\rho)^2} - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x^{-1} - \frac{(1-\rho)^2}{(h+\rho)^3}\partial_x^{-1}\eta^2\partial_x^{-1} + O(\mu).
\end{aligned} \tag{27}$$

Following a similar derivation method to that used for the KdV equation, we obtain the well-known modified Korteweg-de Vries (mKdV) equation:

$$\eta_\tau - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_X - \frac{3c\rho(1+h)^2}{h(h+\rho)^2}\eta^2\eta_X = 0. \tag{28}$$

3.2. Shallow (Lower Layer)-Deep (Upper Layer) Model

We now assume that the upper water layer is relatively deep, with its depth comparable to the wavelength. This implies $h = h^+/h^- \gg 1$ and $h \sim O(\lambda)$. Consistent with the previous discussion, we introduce the small parameter $\mu = 1/\lambda$ and set $h = O(1/\mu)$. We also adopt the following scaling relationships:

$$\partial_x = O(\mu), \partial_t = O(\mu), \eta = O(\mu), \xi = O(1), \rho = O(1), B = O(1/\mu). \tag{29}$$

The scaling relationships above effectively assume strong surface tension, with $B \sim O(1/\mu)$. Under these new scaling conditions, we can derive the asymptotic expansions for the DtN operators

$G^{\pm}$:

$$\begin{aligned}&G^{-}=-\partial_{xx}-\frac{1}{3}\partial_{xxxx}-\partial_{x}\eta\partial_{x}+O\left(\mu^{5}\right),\\&G^{+}=G_{0}^{+}+\partial_{x}\eta\partial_{x}+G_{0}^{+}\eta G_{0}^{+}+O\left(\mu^{5}\right),\\&\left(G^{+}+\rho G^{-}\right)^{-1}=\left(G_{0}^{+}\right)^{-1}+\rho\left(G_{0}^{+}\right)^{-1}\partial_{xx}\left(G_{0}^{+}\right)^{-1}+O(\mu),\end{aligned}\tag{30}$$

Where, G_0^+ is a non-local pseudodifferential operator, with its Fourier transform given by $\widehat{G_0^+}=k\tanh(kh)$. Consequently, the Fourier transform of its inverse operator is $(\widehat{G_0^+})^{-1}=\coth(kh)/k$. By combining equations (14) and (30), and neglecting higher-order small quantities, we obtain:

$$\begin{aligned}&\eta_t+\xi_{xx}+\partial_x\left(\eta\xi_x\right)-\rho\mathcal{K}\left[\xi_{xx}\right]=0,\\&\xi_t+(1-\rho)\eta+\frac{1}{2}\xi_x^2-B\eta_{xx}=0,\end{aligned}\tag{31}$$

Where, $\mathcal{K}$ is a pseudodifferential operator, with its Fourier transform given by $\hat{\mathcal{K}}=k\coth(kh)$. By combining equations (31), and eliminating η while neglecting higher-order terms, we obtain:

$$\xi_{tt}-c^2\xi_{xx}+\partial_x\left(\xi_t\xi_x\right)+\frac{1}{2}\partial_t\left(\xi_x^2\right)+B\xi_{xxxx}+\rho c^2\mathcal{K}\left[\xi_{xx}\right]=0,\tag{32}$$

Where, $c^2=1-\rho$. By introducing the new variables $X=x-ct$ and $\tau=\mu t$, the above equation can be simplified to:

$$H_\tau+\frac{3}{2}HH_X-\frac{\tilde{B}}{2c}H_{XXX}-\frac{\rho c}{2}\mathcal{K}\left[H_X\right]=0,\tag{33}$$

Where, $H=\xi_X$ and $\tilde{B}=\mu B=O(1)$. This equation is known as the Benjamin equation [7]. Additionally, it follows that $\eta=H/c+O(\mu)$. Thus, we have:

$$\eta_\tau+\frac{3c}{2}\eta\eta_X-\frac{\tilde{B}}{2c}\eta_{XXX}-\frac{\rho c}{2}\mathcal{K}\left[\eta_X\right]=0.\tag{34}$$

From equation (34), we observe that, compared to the traditional KdV equation, this equation has an additional term $\rho c\mathcal{K}[\eta_X]/2$.

3.3. Deep (Lower Layer) – Shallow (Upper Layer) Model

Next, following the methods of [16,17], a long-wave model is constructed for deep water (bottom layer) over shallow water (top layer). This model introduces a new small parameter $\mu^2=h/\lambda$ and new scaling relationships:

$$\partial_x=O(1),\partial_t=O(\mu),\eta=O\left(\mu^2\right),\xi=O(\mu),\rho=O(1),B=O(1),h=O\left(\mu^2\right).\tag{35}$$

At this point, the asymptotic expansion of the DtN operator $G^{\pm}$ is given by:

$$\begin{aligned}&G_0^-=D\tanh(hD),\\&G^-=G_0^--\partial_x\eta\partial_x-G_0^-\eta G_0^-+O\left(\mu^4\right),\\&G^+=-h\partial_{xx}+\partial_x\eta\partial_x+O\left(\mu^4\right),\\&\left(G^++\rho G^-\right)^{-1}=\frac{1}{\rho}\left(G_0^-\right)^{-1}+O\left(\mu^2\right).\end{aligned}\tag{36}$$

From equations (65) and (66), we can obtain:

$$\eta_t + \frac{h}{\rho}\xi_{xx} - \frac{1}{\rho}\partial_x(\eta\xi_x) = 0,$$
$$\xi_t + (1-\rho)\eta + \frac{1}{2\rho}\xi_x^2 - B\eta_{xx} = 0. \tag{37}$$

Introducing the variable substitution:

$$\eta = h(1-\Lambda), \xi_x = -\rho\sqrt{h}U, t = \tau/\sqrt{h}, \tag{38}$$

We can obtain:

$$\eta = h(1-\Lambda), \xi_x = -\rho\sqrt{h}U, t = \tau/\sqrt{h},$$
$$\rho(U_\tau + UU_x) - B\Lambda_{xxx} = 0. \tag{39}$$

4. Conclusion

In this study, we systematically investigated the mathematical modelling of nonlinear interfacial waves in two-layer fluid systems. By reformulating the governing equations using an extended Dirichlet–Neumann operator, we derived a class of nonlinear evolution equations that avoid the need to directly solve the Laplace equation, effectively reducing the problem to a system independent of the vertical coordinate. Under long-wave scaling, we expanded the Dirichlet–Neumann operator and employed asymptotic methods to derive various model equations, including the Korteweg–de Vries (KdV), fifth-order KdV, modified KdV (mKdV), and Benjamin equations, as well as strongly nonlinear models. Our results demonstrate that the Dirichlet–Neumann operator expansion provides an efficient and less computationally demanding alternative to traditional approaches for deriving long-wave models in stratified fluid systems.

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